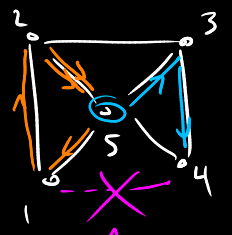
Last Time: Proved the Handshaking Lemma ☺

Defn: Let G be a graph. A walk in G is a sequence $(v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n)$ satisfying:

1. $v_i \in V(G)$ for all $0 \le i \le n$.
2. $e_i \in E(G)$ for all $1 \le i \le n$.
3. e_i has ends v_{i-1} and v_i for all $1 \le i \le n$.

The number n is the length of the walk.

Ex: $G =$



no edge here!

$W_1 = (2)$ ← length-0 walk.

$W_2 = (5, 35, 3, 34, 4)$ ← length 2 walk

$W_3 = (2, 25, 5, 15, 1, 12, 2, 25, 5)$ ← length 4 walk.

Non-example: $(3, 34, 4, 14, 1)$ is NOT a walk in G ← not an edge in G ☹

Prop: Let G be a graph. Define a relation $\sim$ on $V(G)$ by $u \sim v$ when there is a walk connecting u to v. Then $\sim$ is an equivalence relation.

Sketch of Proof: We have for $u, v, w \in V(G)$:

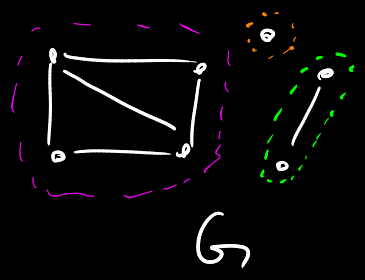
Reflexive: $u \sim u$ holds because (u) is a walk.

Symmetric: Suppose $u \sim v$. So there's a walk from u to v. reverse the walk to obtain a walk from v to u.

Transitive: $u \sim v$ and $v \sim w$, then walking u to v and then v to w yields a walk from u to w.

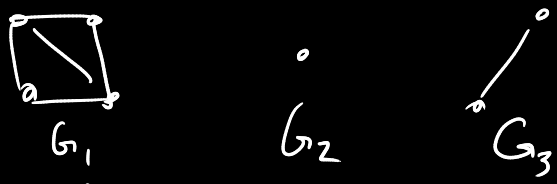
Def: A subgraph of graph $G = (V, E)$ a graph H w/ $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Ex:



G

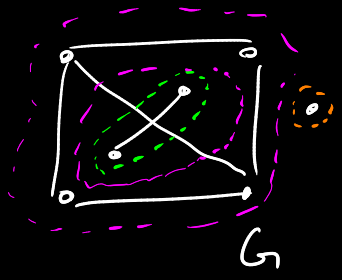
has 3 component subgraphs (i.e. components)



G_1 G_2 G_3

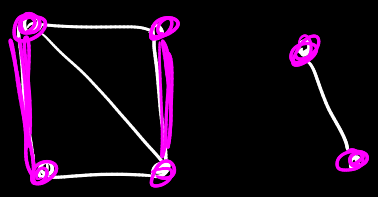
More formally: A component of graph G is a maximal subgraph with exactly one equivalence class under the connection relation.

Ex:



G

Ex:

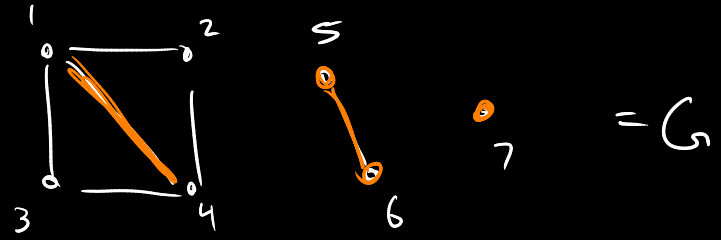


G

H is in pink, and $H \leq G$

"H is a subgraph of G"

Non-Ex:

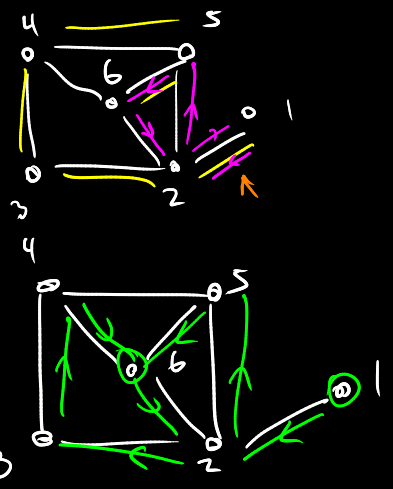


$= G$

$H = (\{5,6,7\}, \{\{4,5\}, \{6\}\})$ ← NOT a graph, so NOT a subgraph.

Defn: [A trail in graph G is a (walk) which does not repeat edges.]

Ex:



Convention: omit edges from walk sequences (for simple graphs only).

$W = (1,2,3,4,5,6)$ yields a trail.

$N = (1,2,5,6,2,1)$ is NOT a trail because it repeats the edge 12...

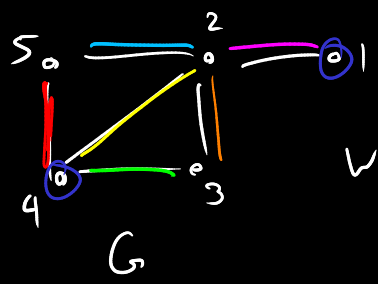
$T = (1,2,5,6,2,3,4,6)$ is a trail.

repeating vertices is OK.

Defn: An Euler trail in graph G is a trail in G which [covers every edge] of G.

Q: Which graphs have Euler trails?

Ex:

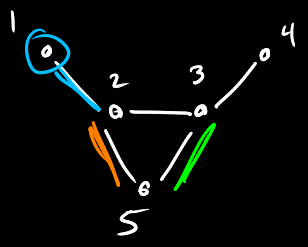


G

$W = (1, 12, 2, 23, 3, 34, 4, 24, 2, 25, 5, 45, 4)$ is an Euler trail in G ☺.

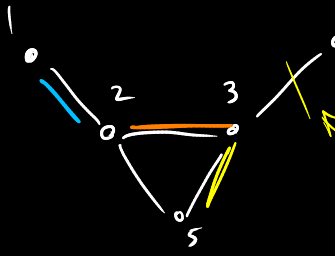
(NB: W here is an "open walk" because its start and end vertices are distinct).

Ex:



$W = (1, 12, 2, 25, 5, 35, 3, ?$

Can't cover both 23 and 34 now...



Can't cross this one ☹

No Euler trail starting at 1.

NB: We could examine every trail in this graph (let's not) to show it doesn't have an Euler trail...

Alternately; If T is an Euler trail in G,

$T = (v_0, e_1, v_1, e_2, v_2, \cdots, v_{n-1}, e_n, v_n)$ where

$E(G) = \{e_1, e_2, \cdots, e_n\}$ with no repeats...

Each vertex of T: v_i has degree $\deg(v_i) = \#\{i \in [n] : v_i \text{ is incident to } e_i\}$

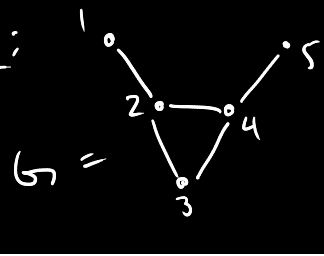
but v_i appears in trail T next to every edge it's incident with...

So v_i (if not equal to v_0 or v_n) has even degree...

This argument yields that every vertex other than v_0, v_n must have even degree...

Point: If G has an Euler trail, then at most two vertices of G have odd degree. *

Ex:

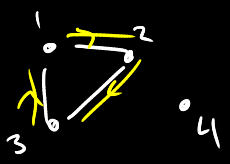


Does not have an Euler trail!

because $\deg(1) = \deg(5) = 1$ and $\deg(2) = \deg(4) = 3$

so G has 4 vertices of odd degree...

Observation: If G has a Euler trail, then every edge of G belongs to the same component of G.

Ex: 

has an Euler trail.

Prop: Let G be a graph. G has an Euler trail if and only if G has at most one component with edges and G has at most two vertices of odd degree. ☺.